Asymptotic expansion of the transition probability for non-symmetric random walks on crystal lattices (Spectral and Scattering Theory and Related Topics)

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1 Introduction

Long time behavior of random walks on graphs has been studied by many authors in various settings. In particular, the central limit theorem (CLT) is one of the most important results in probability theory. As a classical case, let us consider the random walk $\{X_n\}_{n\in\mathbb{N}}$ on the integer lattice $\mathbb{Z}$ given by the sum of iid random variables:

$$X_n = Z_1 + Z_2 + \cdots + Z_n,$$

where $\mathbb{P}(Z_i = 1) = p$ and $\mathbb{P}(Z_i = -1) = 1 - p = q$. Then the central limit theorem asserts that for any $a < b$

$$\lim_{n\to\infty} \mathbb{P}_x \left(a \leq \frac{X_n - m}{\sigma\sqrt{n}} \leq b \right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2t}} dx,$$

where $m = p - q$, the mean of Z_i and $\sigma^2 = 4pq$, the variance of Z_i. In addition, if $m = 0$, that is, the random walk is symmetric, then the transition probability $p(n, x, y) := \mathbb{P}_x(X_n = y)$ has the following integral expression by the well-known Fourier inversion formulra

$$p(n, x, y) = \int_0^1 \cos(n\theta) e^{2\pi\sqrt{-1}\langle\theta,x\rangle} \overline{e^{2\pi\sqrt{-1}\langle\theta,y\rangle}} d\theta.$$

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Using the expansion of $\cos(n\theta)$ (known as the Laplace method), the following long time asymptotics can be obtained:

Theorem 1.1 (Local central limit theorem (LCLT))

$$\lim_{n\to\infty} \sqrt{2n\pi} \left| p(n,x,y) - e^{-\frac{|x-y|^2}{2n}} \right| = 0.$$

Theorem 1.2 (Asymptotic expansion)

$$p(n,x,y) \sim \frac{1}{\sqrt{2n\pi}} e^{-\frac{|x-y|^2}{2n}} \left(1 + c_1(x,y)n^{-1} + c_2(x,y)n^{-2} \cdots c_k(x,y)n^{-k}\right).$$

Here these limits are taken uniformly for x, y in some domains. See for instance, [9], [10] and [14] for the detail.

As Spitzer has mentioned in [10], the periodicity of $\mathbb{Z}^d$ plays a crucial role to obtain the asymptotics. This motivated Shirai and Sunada to study the long time asymptotics of a symmetric random walk on a *crystal lattice*, an abelian covering of a finite graph.

They emphasized in their articles ([3], [4], [5], [6], [7], [11], [12], [13]) that the long time behavior of the random walk must not depend on the choice of the realization. In this geometric point of view, they found that a canonical realization with a flat metric is naturally appeared in the asymptotics called a *standard realization.* Moreover, Sunada presented in [12] the local central limit theorem for the non-symmetric random walks on crystal lattices. In the proof, the notion of the modified standard realization of the graph into the corresponding continuous space plays a crucial role.

In this exposition, we give an outline of the proof of the asymptotic expansion of the transition probability of the non-symmetric random walks on crystal lattices which is recently proved in [1]. See also [2] for the explicit calculation on the triangular lattice.

2 Discrete harmonic analysis on crystal lattices

We review the general facts about the discrete harmonic analysis on crystal lattices.

First of all, we give the definition of the crystal lattice. An oriented, locally finite connected graph $X = (V, E)$ is called a crystal lattice if there exists an abelian group Γ acting on X freely and its quotient $X_0 = \Gamma\backslash X$ is a finite graph. In other words, X is the abelian cover of a finite graph X_0 with the covering transformation group Γ (see Figure 1). Without loss of generality, we always assume that Γ is torsion-free ($\Gamma \simeq \mathbb{Z}^d$ for some d) by changing the quotient X_0.

For an oriented edge $e \in E$, the origin and the terminus of e are denoted by $o(e)$ and $t(e)$, respectively. The inverse edge of $e \in E$ is denoted by $\bar{e}$. Let $E_x = \{e \in E \mid o(e) = x\}$

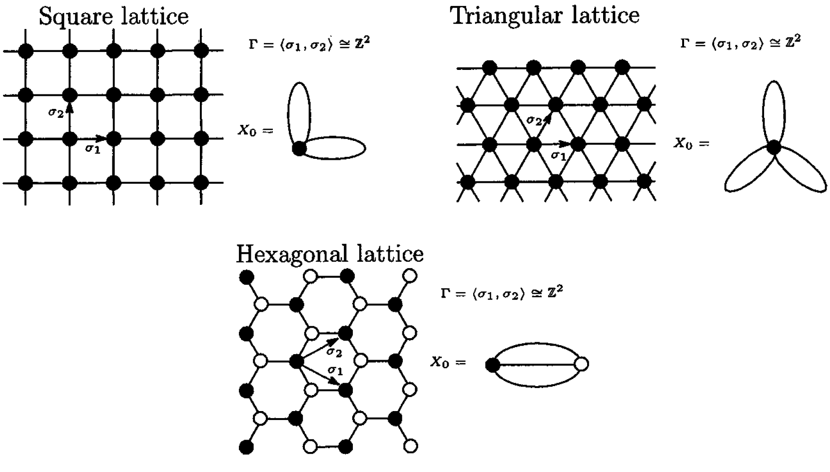

Figure 1: Crystal lattices

be the set of edges with $o(e) = x \in V$. We consider a random walk on X given by a Γ-invariant transition probability $p : E \to [0,1]$ with

$$\sum_{e \in E_x} p(e) = 1, \quad p(e) + p(\overline{e}) > 0 \quad (e \in E).$$

The transition operator L is an operator acting on functions on X defined by

$$Lf(x) := \sum_{e \in E_x} p(e) f(t(e)).$$

Then the n-step transition probability $p(n,x,y)$ from x to y is given by

$$p(n,x,y) := \sum_{(e_1, e_2, \ldots, e_n)} p(e_1)p(e_2)\cdots p(e_n),$$

where the sum is taken over all n-length paths $(e_1, e_2, \ldots, e_n)$from x to y. We mention

$$L^n f(x) = \sum_{y \in V} p(n,x,y) f(y) \qquad (x \in V).$$

If, in addition, there exists a positive function $m : V \to \mathbb{R}$ such that

$$p(e)m(o(e)) = p(\overline{e})m(t(e)) \qquad (e \in E),$$

then the random walk is said to be *symmetric* (or *reversible*), and the function m is called a *reversible measure* for the random walk. Note that m is uniquely determined up to a constant multiple. The most canonical symmetric random walk is the simple random walk with the transition probability given by $p(e) = (\deg o(e))^{-1}$ $(e \in E)$.

Now let us consider the discrete spectral analysis on X_0. By the Γ-periodicity of the random walk on X, the corresponding random walk on X_0 can be defined. We always assume that the random walk on X_0 is *irreducible*, that is, for every $x, y \in V_0$, there exists some $n = n(x,y) \in \mathbb{N}$ such that $p(n,x,y) > 0$. We note that the irreducibility on X_0 holds on X. However, the converse does not in general. By the Perron–Frobenius theorem, there exists a unique positive function $m : V_0 \to \mathbb{R}$, called the invariant probability measure, satisfying $\sum_{x \in V_0} m(x) = 1$ and

$$m(x) = {}^tLf(x), \tag{2.1}$$

where ${}^tLf(x) = \sum_{e \in (E_0)_x} p(\overline{e})f(t(e))$, the transposed operator of L. It means that the invariant measure m is an eigenfunction of ${}^tL$ for the maximal positive eigenvalue 1.

We put

$$\widetilde{m}(e) = m(o(e))p(e) \qquad (e \in E_0).$$

When $\widetilde{m}(e) = \widetilde{m}(\overline{e})$, the random walk is said to be $(m\text{-})$*symmetric*.

We consider the 0-chain group

$$C_0(X_0, \mathbb{R}) = \Big\{ \sum_{x \in V_0} a_x x \,\Big|\, a_x \in \mathbb{R} \Big\}$$

and the 1-chain group

$$C_1(X_0, \mathbb{R}) = \Big\{ \sum_{e \in E_0} a_e e \,\Big|\, a_e \in \mathbb{R} \Big\},$$

where the relation $\overline{e} = -e$ is imposed for $e \in E_0$. The boundary operator $\partial : C_1(X_0, \mathbb{R}) \to C_0(X_0, \mathbb{R})$ is defined by $\partial(e) := t(e) - o(e)$. The first homology group $\mathrm{H}_1(X_0, \mathbb{R})$ is the kernel $\mathrm{Ker}(\partial) \subset C_1(X_0, \mathbb{R})$. $\mathrm{H}_1(X_0, \mathbb{Z})$ is also defined by replacing $\mathbb{R}$ by $\mathbb{Z}$.

We define the 0-cochain group

$$C^0(X_0, \mathbb{R}) := \{ f : V_0 \to \mathbb{R} \}$$

with the inner product

$$\langle f_1, f_2 \rangle_0 = \sum_{x \in V_0} f_1(x) f_2(x) \qquad (f_1, f_2 \in C^0(X_0, \mathbb{R})),$$

and the 1-cochain group

$$C^1(X_0, \mathbb{R}) := \{ \omega : E_0 \to \mathbb{R} \,|\, \omega(\overline{e}) = -\omega(e) \}$$

with the inner product

$$\langle \omega_1, \omega_2 \rangle_1 = \frac{1}{2} \sum_{e \in E_0} \omega_1(e) \omega_2(e) \qquad (\omega_1, \omega_2 \in C^1(X_0, \mathbb{R})).$$

A 1-cochain is occasionally called a 1-form on X_0.

We define the difference operator $d : C^0(X_0, \mathbb{R}) \to C^1(X_0, \mathbb{R})$ by

$$df(e) := f(t(e)) - f(o(e)) \qquad (e \in E_0),$$

and the first cohomology group $\mathrm{H}^1(X_0, \mathbb{R}) := C^1(X_0, \mathbb{R})/\mathrm{Im}(d)$. Note that $\mathrm{H}^1(X_0, \mathbb{R})$ is the dual of the first homology group $\mathrm{H}_1(X_0, \mathbb{R})$.

We define the homological direction by

$$\gamma_p := \sum_{e \in E_0} \widetilde{m}(e)e \in C_1(X_0, \mathbb{R}).$$

We observe that $\partial\gamma_p = 0$ and hence, $\gamma_p \in \mathrm{H}_1(X_0, \mathbb{R})$. We note that $\gamma_p = 0$ if and only if p gives a symmetric random walk, i.e., $\widetilde{m}(e) = \widetilde{m}(\overline{e})$. A 1-form ω is said to be *modified harmonic* if

$$\delta_p\omega(x) + \langle\gamma_p, \omega\rangle = 0 \qquad (x \in V_0), \tag{2.2}$$

where $(\delta_p\omega)(x) := -\sum_{e \in (E_0)_x} p(e)\omega(e)$ and $\langle\gamma_p, \omega\rangle := {}_{C_1(X_0,\mathbb{R})}\langle\gamma_p, \omega\rangle_{C^1(X_0,\mathbb{R})}$ is constant as a function on V_0. We denote by $\mathcal{H}^1(X_0)$ the set of modified harmonic 1-forms, and equip $\mathcal{H}^1(X_0)$ with the inner product

$$\langle\!\langle \omega_1, \omega_2 \rangle\!\rangle := \sum_{e \in E_0} \omega_1(e)\omega_2(e)\widetilde{m}(e) - \langle\gamma_p, \omega_1\rangle\langle\gamma_p, \omega_2\rangle \qquad (\omega_1, \omega_2 \in \mathcal{H}^1(X_0)). \tag{2.3}$$

Then the corresponding norm $\|\cdot\|$ is given by

$$\|\omega\|^2 := \langle\!\langle \omega, \omega \rangle\!\rangle = \sum_{e \in E_0} |\omega(e)|^2\widetilde{m}(e) - \langle\gamma_p, \omega\rangle^2 \qquad (\omega \in \mathcal{H}^1(X_0)).$$

By the discrete Hodge–Kodaira theorem (cf. [7, Lemma 5.2]), we may identify $\mathrm{H}^1(X_0, \mathbb{R})$ and $\mathrm{H}^1(X_0, \mathbb{Z})$ with $\mathcal{H}^1(X_0)$ and

$$\Big\{\omega \in \mathcal{H}^1(X_0) \Big| \int_c \omega := \sum_{i=1}^{n} \omega(e_i) \in \mathbb{Z} \text{ for every closed path } c = (e_1, \dots, e_n) \text{ in } X_0\Big\},$$

respectively. Using this identification, we obtain an inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathrm{H}^1(X_0, \mathbb{R})$.

We denote by $\pi : X \to X_0$ the covering map, and by $\rho : \mathrm{H}_1(X_0, \mathbb{Z}) \to \Gamma$ the surjective homomorphism associated with the covering map π. We extend ρ to the surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \to \Gamma \otimes \mathbb{R}$. Then we may consider the injective linear map ${}^t\rho_{\mathbb{R}} : \mathrm{Hom}(\Gamma, \mathbb{R}) \to \mathrm{H}^1(X_0, \mathbb{R})$ by

$${}^t\rho_{\mathbb{R}} : \omega \in \mathrm{Hom}(\Gamma, \mathbb{R}) \mapsto {}^t\rho_{\mathbb{R}}(\omega)(\cdot) := \omega(\rho_{\mathbb{R}}(\cdot)) \in \mathrm{H}^1(X_0, \mathbb{R}),$$

where $\mathrm{Hom}(\Gamma, \mathbb{R})$ denotes the linear space of homomorphisms of Γ into $\mathbb{R}$. Using the maps ${}^t\rho_{\mathbb{R}}$ and $\rho_{\mathbb{R}}$, we identify $\mathrm{Hom}(\Gamma, \mathbb{R})$ with the subspace $\mathrm{Image}({}^t\rho_{\mathbb{R}})$ in $\mathrm{H}^1(X_0, \mathbb{R})$ and

$\Gamma\otimes\mathbb{R}$ with the quotient linear subspace of $\mathrm{H}_1(X_0,\mathbb{R})$. We denote ${}^t\rho_{\mathbb{R}}(\omega)\in \mathrm{H}^1(X_0,\mathbb{R})$ by the same symbol ω for brevity. We restrict the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on $\mathrm{H}^1(X_0,\mathbb{R})$ to the subspace $\mathrm{Hom}(\Gamma,\mathbb{R})$, and then take up the dual inner product $\langle\cdot,\cdot\rangle_{alb}$ on $\Gamma\otimes\mathbb{R}$. The flat metric on $\Gamma\otimes\mathbb{R}$ induced from this inner product is called the *Albanese metric* and is denoted by g_0. This procedure is summarized in the following diagram:

$$\begin{array}{ccc} (\Gamma\otimes\mathbb{R},g_0) & \xleftarrow{\rho_{\mathbb{R}}} & \mathrm{H}_1(X_0,\mathbb{R}) \\ \updownarrow \text{dual} & & \updownarrow \text{dual} \\ \mathrm{Hom}(\Gamma,\mathbb{R}) & \xrightarrow{{}^t\rho_{\mathbb{R}}} & \mathrm{H}^1(X_0,\mathbb{R})\cong \big(\mathcal{H}^1(X_0),\langle\!\langle\cdot,\cdot\rangle\!\rangle\big) \end{array}$$

We write Alb^{Γ} for $(\Gamma\otimes\mathbb{R}/\Gamma\otimes\mathbb{Z},g_0)$, and call it the Γ-*Albanese torus* associated with (X,Γ).

Now we realize X in $\Gamma\otimes\mathbb{R}$ equipped with the Albanese metric g_0 in a standard way. A (piecewise linear) map $\Phi: X\to\Gamma\otimes\mathbb{R}$ is said to be a *periodic realization* of X if it satisfies

$$\Phi(\sigma x)=\Phi(x)+\sigma\otimes 1 \qquad (x\in X,\ \sigma\in\Gamma).$$

We may define a special periodic realization $\Phi_0: X\to\Gamma\otimes\mathbb{R}$ by $\Phi_0(x_*)=\mathbf{0}$ for a fixed base point $x_*\in V$ and

$$_{\mathrm{Hom}(\Gamma,\mathbb{R})}\big\langle\omega,\Phi_0(x)\big\rangle_{\Gamma\otimes\mathbb{R}}=\int_{x_*}^{x}\widetilde{\omega} \qquad (\omega\in\mathrm{Hom}(\Gamma,\mathbb{R})), \tag{2.4}$$

where $\widetilde{\omega}$ is the lift of $\omega={}^t\rho_{\mathbb{R}}(\omega)\in\mathrm{H}^1(X_0,\mathbb{R})$ to X. Here

$$\int_{x_*}^{x}\widetilde{\omega}=\int_c\widetilde{\omega}:=\sum_{i=1}^{n}\widetilde{\omega}(e_i)$$

for a path $c=(e_1,\dots,e_n)$ with $o(e_1)=x_*$ and $t(e_n)=x$. It should be noted that this line integral does not depend on the choice of a path c.

One of the special properties of Φ_0 is that it is a vector-valued *modified-harmonic function* on X in the sense that

$$L\Phi_0(x)-\Phi_0(x)=\rho_{\mathbb{R}}(\gamma_p) \qquad (x\in V). \tag{2.5}$$

Indeed, for every $\omega={}^t\rho_{\mathbb{R}}(\omega)\in\mathrm{H}^1(X_0,\mathbb{R})$, the modified harmonicity (2.2), Γ-invariance of the transition probability p and the identity (2.4) imply

$$\begin{aligned}
_{\mathrm{Hom}(\Gamma,\mathbb{R})}\big\langle\omega,L\Phi_0(x)-\Phi_0(x)\big\rangle_{\Gamma\otimes\mathbb{R}} &= \sum_{e\in E_x}p(e)\,{}_{\mathrm{Hom}(\Gamma,\mathbb{R})}\big\langle\omega,\Phi_0(t(e))-\Phi_0(o(e))\big\rangle_{\Gamma\otimes\mathbb{R}}\\
&= \sum_{e\in E_x}p(e)\widetilde{\omega}(e)\\
&= \sum_{e\in(E_0)_{\pi(x)}}p(e)\omega(e)\\
&= -(\delta_p\omega)(\pi(x))\\
&= \langle\gamma_p,\omega\rangle = {}_{\mathrm{Hom}(\Gamma,\mathbb{R})}\big\langle\omega,\rho_{\mathbb{R}}(\gamma_p)\big\rangle_{\Gamma\otimes\mathbb{R}} \qquad (x\in V).
\end{aligned}$$

A periodic realization $\Phi : X \to \Gamma \otimes \mathbb{R}$ satisfying (2.5) is said to be *modified harmonic.* Note that a modified harmonic realization is uniquely determined up to translation.

If we equip $\Gamma \otimes \mathbb{R}$ with the Albanese metric g_0, then we call the map $\Phi_0 : X \to (\Gamma \otimes \mathbb{R}, g_0)$ the *modified standard realization* of X. We readily check that the piecewise linear interpolation of Φ_0 by line segments descends to a piecewise geodesic map $\Phi_0 : X_0 \to \mathrm{Alb}^\Gamma$. We call Φ_0 the *Albanese map* associated with (X, Γ). Namely, standard realization is a lift of the Albanese map.

In [12], Sunada presented the local central limit theorem (LCLT) for non-symmetric random walk on crystal lattices stated as follows: Let

$$K := \mathrm{g.c.d.}\{\mathrm{n} \in \mathbb{N}; \mathrm{p}(\mathrm{n}, \mathrm{x}, \mathrm{x}) \neq 0\}$$

be the period of the random walk and $V = \coprod_{k=0}^{K-1} A_k$ the K-partition of V.

Theorem 2.1 **([12])** *Let $x \in A_i$, $y \in A_j$. If $n = Kl + j - i$,*

$$p(n, x, y)m(y)^{-1} \sim \frac{K\mathrm{vol}(\mathrm{Alb}^\Gamma)}{(2\pi n)^{d/2}} e^{\left(-\frac{\left|\Phi_0(x) - \Phi_0(y) - n\rho_{\mathbb{R}}(\gamma_p)\right|_{g_0}^2}{2n}\right)}.$$

Otherwise $p(n, x, y) = 0$, where $\mathrm{vol}(\mathrm{Alb}^\Gamma)$ *is the volume of* Alb^Γ *with the Albanse metric* g_0.

3 Asymptotic expansion of the transition probability

Recall that $X = (V, E)$ is a crystal lattice in which covering transformation group Γ is a torsion free abelian group of rank d and torsion free. Let K be the period of the random walk on X. We note that we can retake the covering transformation group so that the period of the corresponding random walk on X_0 has the same period K.

3.1 Twisted transition operators

We first review some basic results on the twisted transition operators studied in [3, 4, 8]. Let $\widehat{\mathrm{H}}_1(X_0, \mathbb{Z})$ be the group of unitary characters of $\mathrm{H}_1(X_0, \mathbb{Z})$. We identify $\widehat{\mathrm{H}}_1(X_0, \mathbb{Z})$ with the Jacobian torus

$$J(X_0) := \mathrm{H}^1(X_0, \mathbb{R})/\mathrm{H}^1(X_0, \mathbb{Z})$$

by the mapping

$$\mathrm{H}^1(X_0, \mathbb{R}) \ni \omega \mapsto \chi_\omega \in \widehat{\mathrm{H}}_1(X_0, \mathbb{Z}),$$

where

$$\chi_\omega(\sigma) := \exp\left(2\pi\sqrt{-1}\int_{c_\sigma} \omega\right) \qquad (\sigma \in \Gamma)$$

and c_σ is a closed path in X_0 satisfying $\rho([c_\sigma]) = \sigma$. We equip a flat metric on $J(X_0)$ induced by the metric (2.3) on $\mathrm{H}^1(X_0,\mathbb{R})(\cong \mathcal{H}^1(X_0))$.

Let $\widehat{\Gamma}$ be the group of unitary characters of the covering transformation group Γ. By the above mapping, we can also identify $\widehat{\Gamma}$ with the Γ-Jacobian torus

$$\mathrm{Jac}^\Gamma := \mathrm{Hom}(\Gamma,\mathbb{R})/\mathrm{Hom}(\Gamma,\mathbb{Z}).$$

The canonical surjective homomorphism $\rho : \mathrm{H}_1(X_0,\mathbb{Z}) \to \Gamma$ gives rise to an injective homomorphism Jac^Γ into $J(X_0)$. We regard Jac^Γ as the flat torus with the metric induced by that on $J(X_0)$. The tangent space $T_{\mathbf{1}}\widehat{\Gamma}$ at the trivial character $\mathbf{1}$ coincides with $\{\omega \in \mathrm{H}^1(X_0,\mathbb{R}) | \ \chi_\omega \in \widehat{\Gamma}\}$, and it is identified with $\mathrm{Hom}(\Gamma,\mathbb{R})$ (see Figure 2). Since the lattice

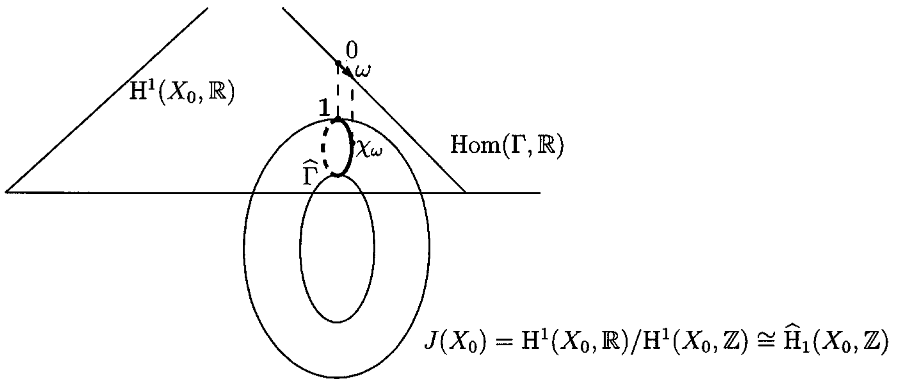

Figure 2: $\widehat{\Gamma} \subset J(X_0)$ and $\mathrm{Hom}(\Gamma,\mathbb{R}) \subset \mathrm{H}^1(X_0,\mathbb{R})$.

group $\Gamma\otimes\mathbb{Z}$ in $\Gamma\otimes\mathbb{R}$ and the lattice group $\mathrm{Hom}(\Gamma,\mathbb{Z})$ in $\mathrm{Hom}(\Gamma,\mathbb{R})$ are dual each other, we observe that the Γ-Albanese torus $\mathrm{Alb}^\Gamma = (\Gamma\otimes\mathbb{R}/\Gamma\otimes\mathbb{Z}, g_0)$ is the dual flat torus of Jac^Γ, and hence $\mathrm{vol}(\widehat{\Gamma}) = \mathrm{vol}(\mathrm{Jac}^\Gamma) = \mathrm{vol}(\mathrm{Alb}^\Gamma)^{-1}$.

To analyze the n-step transition probability $p(n,x,y)$ for the random walk on the crystal lattice $X = (V,E)$, we introduce the *twisted transition operator* L_χ for a unitary character $\chi \in \widehat{\Gamma}$. For each $\chi \in \widehat{\Gamma}$, we consider the $|V_0|$-dimensional inner product space

$$\ell^2_\chi = \{f : X \to \mathbb{C} | \ f(\sigma x) = \chi(\sigma)f(x) \text{ for } \sigma \in \Gamma\}$$

with the inner product

$$\langle f,g\rangle_\chi = \sum_{x\in\mathcal{F}} f(x)\overline{g(x)},$$

where $\mathcal{F} \subset V$ is a fundamental domain of X for Γ. We note that the inner product is independent of the choice of a fundamental domain $\mathcal{F}$.

As the transition operator L and its transpose ${}^tL$ preserve ℓ^2_χ (see [8]), we define the *twisted transition operator* $L_\chi : \ell^2_\chi \to \ell^2_\chi$ and its transposed operator ${}^tL_\chi : \ell^2_\chi \to \ell^2_\chi$ by the restriction of L and ${}^tL$, respectively. For the trivial character $\chi = \mathbf{1}$, $(L_{\mathbf{1}}, \ell^2_{\mathbf{1}})$ and $({}^tL_{\mathbf{1}}, \ell^2_{\mathbf{1}})$ are identified with $(L, \ell^2(X_0))$ and $({}^tL, \ell^2(X_0))$, respectively. The family $\{L_\chi\}_{\chi\in\widehat{\Gamma}}$ gives rise to the direct integral decomposition

$$(L, \ell^2(X)) = \int_{\widehat{\Gamma}}^{\oplus} (L_\chi, \ell^2_\chi) d\chi,$$

where $d\chi$ denotes the normalized Haar measure on $\widehat{\Gamma}$. As in [8, Section 7], this decomposition implies an integral expression of the n-step transition probability

$$p(n, x, y) = \int_{\widehat{\Gamma}} \langle L^n_\chi f_y, f_x \rangle_\chi d\chi \qquad (n \in \mathbb{N},\ x, y \in V), \tag{3.1}$$

where $f_x \in \ell^2_\chi$ is the modified delta function defined by

$$f_x(z) := \begin{cases} \chi(\sigma) & \text{if} \quad z = \sigma x, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from $\langle L^n_\chi f_y, f_x \rangle_\chi = 0$ $(x \in A_i, y \in A_j, n \neq Kl + j - i)$ that $p(n, x, y) = 0$.

From this general viewpoint, let us see the case of the square lattice graph $\mathbb{Z}^d$. In this case, we note that $X = \Gamma = \mathbb{Z}^d$ and $|V_0| = 1$. Moreover,

$$\hat{\Gamma} = \left\{\chi(\sigma) = e^{2\pi\sqrt{-1}\langle\theta,\sigma\rangle} \mid \theta \in [0,1)^d\right\} \simeq \mathbb{T}^d.$$

Then we obtain

$$\ell^2_\chi = \ell^2_\theta = \operatorname{span}\{e^{2\pi\sqrt{-1}\langle\theta,\cdot\rangle}\}$$

and hence, for $f \in \ell^2(X)$,

$$f(x) = \int_{\theta\in\mathbb{T}^d} a(\theta) e^{2\pi\sqrt{-1}\langle\theta,x\rangle} d\theta,$$

which is nothing but the Fourier inversion formula.

By virtue of the Perron–Frobenius theorem for the random walk with period K, the twisted transition operator L_χ has K-simple maximum eigenvalues $\mu_0(\chi), \ldots, \mu_{K-1}(\chi)$ with eigenfunctions satisfying

$$L_\chi \phi_{k,\chi} = \mu_k(\chi)\phi_{k,\chi}, \quad {}^tL_\chi \psi_{k,\chi} = \overline{\mu_k(\chi)}\psi_{k,\chi},$$

$$\langle \phi_{k,\chi}, \phi_{k,\chi} \rangle_\chi = \langle \phi_{k,\chi}, \psi_{k,\chi} \rangle_\chi = 1.$$

Then we note that

$$\mu_0(\mathbf{1}) = 1, \ \mu_k(\chi) = e^{(\frac{2k\pi\sqrt{-1}}{K})}\mu_0(\chi)$$

and, for $x \in A_i$,

$$\phi_{k,\chi}(x) = e^{(\frac{2ki\pi\sqrt{-1}}{K})}\phi_{0,\chi}(x),\ \psi_{k,\chi}(x) = e^{(\frac{2ki\pi\sqrt{-1}}{K})}\psi_{0,\chi}(x). \tag{3.2}$$

We also note that

$$\|L_\chi\| \leq 1 \quad (|\mu_k(\chi)| \leq 1),$$
$$\|L_\chi\| = 1 \Leftrightarrow \chi = \mathbf{1} \quad (|\mu_k(\chi)| = 1 \Leftrightarrow \chi = \mathbf{1})$$

and

$$\ell^2_\chi = \oplus_{k=0}^{K-1}\langle\phi_{k,\chi}\rangle + V_\chi,$$

where $\|L_\chi f_V\| < (1-\epsilon)\|f_V\|$ for $f_V \in V_\chi$. By (3.1), we obtain

$$p(n,x,y) = L^n\delta_y(x) \tag{3.3}$$
$$= \int_{\widehat{\Gamma}}\Big(\sum_{k=0}^{K-1}\mu_k(\chi)^n\phi_{k,\chi}(x)\overline{\psi_{k,\chi}(y)} + \langle L^n_\chi\{(f_y)_{\mathcal{V}_\chi}\}, f_x\rangle_\chi\Big)d\chi. \tag{3.4}$$

Since L_χ preserves $\mathcal{V}_\chi$, and $\|L_\chi|_{\mathcal{V}_\chi}\| < 1-\varepsilon$ for some $\varepsilon > 0$ uniformly in $\chi \in \widehat{\Gamma}$ (see [3]), we have

$$\Big|\int_{\widehat{\Gamma}}\langle L^n_\chi\{(f_y)_{\mathcal{V}_\chi}\}, f_x\rangle_\chi d\chi\Big| \leq C(1-\varepsilon)^n \tag{3.5}$$

for some positive constant C independent of x and y. Therefore substituting (3.2) into (3.4), we obtain

$$p(n,x,y) \sim \int_{\widehat{\Gamma}}\sum_{k=0}^{K-1} e^{\left(\frac{2k(n-i-j)\pi\sqrt{-1}}{K}\right)}\mu_0(\chi)^n\phi_{0,\chi}(x)\overline{\psi_{0,\chi}(y)}d\chi.$$

If $n = Kl + j - i$, we conclude

$$p(n,x,y) \sim K\int_{\widehat{\Gamma}}\mu_0(\chi)^n\phi_{0,\chi}(x)\overline{\psi_{0,\chi}(y)}d\chi. \tag{3.6}$$

Using a unitary map $\ell^2(X_0)$ with $\ell^2_{\chi_\omega}$ given by

$$f \mapsto s_\omega(x)f(x) = e^{2\pi\sqrt{-1}\langle\omega,\Phi_0(x)\rangle}f(x),$$

we can rewrite $\phi_{0,\chi}$ and $\psi_{0,\chi}$ as

$$\phi_{0,\chi_\omega}(x) = s_\omega(x)\phi_\omega(x), \quad \psi_{0,\chi_\omega}(x) = s_\omega(x)\psi_\omega(x),$$

where ϕ_ω and ψ_ω is the eigenfunction of the *Harper operator* H_ω and its adjoint H^*_ω acting on $\ell^2(X_0)$ defined by

$$H_\omega f(x_0) := \sum_{e\in(E_0)_{x_0}} p(e)\exp(2\pi\sqrt{-1}\omega(e))f(t(e)) \qquad (x_0 \in V_0),$$
$$H^*_\omega f(x_0) := \sum_{e\in(E_0)_{x_0}} p(\overline{e})\exp(2\pi\sqrt{-1}\omega(e))f(t(e)) \qquad (x_0 \in V_0),$$

respectively. Putting $\mu_0(\chi) = e^{-\lambda(\omega)}$, we obtain

$$\begin{aligned}
&\mu_0(\chi_\omega)^n \phi_{0,\chi_\omega}(x)\overline{\psi_{0,\chi_\omega}(y)} \\
&= e^{-n\lambda(\omega)}\phi_\omega(\pi(x))\overline{\psi_\omega(\pi(y))}\exp\Big(2\pi\sqrt{-1}\int_y^x \tilde{\omega}\Big) \\
&= e^{-n\lambda(\omega)}\phi_\omega(\pi(x))\overline{\psi_\omega(\pi(y))}\exp\Big(-2\pi\sqrt{-1}\langle\omega, \Phi_0(y) - \Phi_0(x)\rangle\Big) \qquad (3.7)
\end{aligned}$$

for $x \in A_i, y \in A_j, n = Kl + j - i$ and sufficiently small $\omega \in \mathrm{Hom}(\Gamma, \mathbb{R})$. Substituting (3.7) into (3.6), we have

$$\begin{aligned}
p(n,x,y) &\approx K\int_{\hat{\Gamma}} \mu_0(\chi)^n\phi_{0,\chi}(x)\overline{\psi_{0,\chi}(y)}d\chi \\
&= K\int_{\mathrm{Hom}(\Gamma,\mathbb{R})\subset\mathcal{H}^1(X_0)} e^{-n\lambda(\omega)}\phi_\omega(\pi(x))\overline{\psi_\omega(\pi(y))}e^{\left(-2\pi\sqrt{-1}\langle\omega,\Phi_0(y)-\Phi_0(x)\rangle\right)}\frac{d\omega}{\mathrm{vol}(\mathrm{Jac}^\Gamma)} \\
&= \frac{K\mathrm{vol}(\mathrm{Alb}^\Gamma)}{n^{\frac{d}{2}}}\int_{\mathrm{Hom}(\Gamma,\mathbb{R})\subset\mathcal{H}^1(X_0)} e^{-n\lambda(\frac{\omega}{\sqrt{n}})}\phi_{\frac{\omega}{\sqrt{n}}}(\pi(x))\overline{\psi_{\frac{\omega}{\sqrt{n}}}(\pi(y))}e^{\left(-2\pi\sqrt{-1}\langle\frac{\omega}{\sqrt{n}},\Phi_0(y)-\Phi_0(x)\rangle\right)}d\omega.
\end{aligned}$$

In order to obtain the desired long time asymptotics of $p(n,x,y)$, we need derivatives of $\lambda(\omega)$, ϕ_ω and ψ_ω.

Lemma 3.1 *For $\omega \in \mathrm{Hom}(\Gamma,\mathbb{R})$, Let $\lambda(t) := \lambda(t\omega)$. For any $k \in \mathbb{N}$, changing eigensections s_ω, if necessary, for any $1 \le i \le k$, the i-th derivatives $\lambda^{(i)}(0)$ are the i-th order real coefficient homogeneous polynomials of $\sqrt{-1}\omega$. In particular,*

$$\begin{aligned}
\lambda(0) &= 0, \\
\lambda'(0) &= -2\pi\sqrt{-1}\langle\gamma_p,\omega\rangle, \\
\lambda''(0) &= 4\pi^2\Big(\sum_{e\in E_0} p(e)\omega(e)^2 m(o(e)) - \langle\gamma_p,\omega\rangle^2\Big) = 4\pi^2\|\omega\|^2, \\
\lambda^{(3)}(0) &= 8\pi^3\sqrt{-1}\sum_{e\in E_0} p(e)\omega(e)^3 m(o(e)) - 24\pi^2\sqrt{-1}\|\omega\|^2\langle\gamma_p,\omega\rangle - 8\pi^3\sqrt{-1}\langle\gamma_p,\omega\rangle^3 \\
&\quad -6\pi\sqrt{-1}|V_0|^{1/2}\sum_{e\in E_0} p(e)\omega(e)d\phi_0''(e)m(o(e)), \\
\lambda^{(4)}(0) &= -16\pi^4\sum_{e\in E_0} p(e)\omega(e)^4 m(o(e)) + 48\pi^4\|\omega\|^4 \\
&\quad +64\pi^4\langle\gamma_p,\omega\rangle\sum_{e\in E_0} p(e)\omega(e)^3 m(o(e)) - 96\pi^4\langle\gamma_p,\omega\rangle^2\|\omega\|^2 - 48\pi^4\langle\gamma_p,\omega\rangle^4 \\
&\quad -48\pi^2|V_0|^{1/2}\langle\gamma_p,\omega\rangle\sum_{e\in E_0} p(e)\omega(e)d\phi_0''(e)m(o(e)) \\
&\quad +24\pi^2|V_0|^{1/2}\sum_{e\in E_0} p(e)\omega(e)^2\Big(\phi_0''(t(e)) - \sum_{z\in V_0}\phi_0''(z)m(z)\Big)m(o(e)) \\
&\quad -8\pi\sqrt{-1}|V_0|^{1/2}\sum_{e\in E_0} p(e)\omega(e)d\phi_0^{(3)}(e)m(o(e)).
\end{aligned}$$

Remark 3.2 *Differentiating both sides of $H_t^* \psi_t = \exp(-\overline{\lambda_\omega(t)})\psi_t$ four times in t at $t=0$, we also obtain*

$$\begin{aligned}\lambda^{(3)}(0) &= 8\pi^3\sqrt{-1}\sum_{e\in E_0} p(e)\omega(e)^3 m(o(e)) - 24\pi^2\sqrt{-1}\|\omega\|^2\langle\gamma_p,\omega\rangle - 8\pi^3\sqrt{-1}\langle\gamma_p,\omega\rangle^3 \\ &\quad +12\pi^2|V_0|^{-1/2}\sum_{e\in E_0} p(e)\omega(e)^2\overline{\psi_0'(o(e))},\end{aligned}$$

and

$$\begin{aligned}\lambda^{(4)}(0) &= -16\pi^4\sum_{e\in E_0} p(e)\omega(e)^4 m(o(e)) + 48\pi^4\|\omega\|^4 \\ &\quad +64\pi^4\langle\gamma_p,\omega\rangle\sum_{e\in E_0} p(e)\omega(e)^3 m(o(e)) - 96\pi^4\langle\gamma_p,\omega\rangle^2\|\omega\|^2 \\ &\quad -48\pi^4\langle\gamma_p,\omega\rangle^4 - 96\pi^3\sqrt{-1}|V_0|^{-1/2}\langle\gamma_p,\omega\rangle\sum_{e\in E_0} p(e)\omega(e)^2\overline{\psi_0'(o(e))} \\ &\quad +32\pi^3\sqrt{-1}|V_0|^{-1/2}\sum_{e\in E_0} p(e)\omega(e)^3\overline{\psi_0'(o(e))} \\ &\quad +24\pi^2|V_0|^{-1/2}\sum_{e\in E_0} p(e)\omega(e)^2\Big(\psi_0''(o(e)) - \sum_{z\in V_0}\psi_0''(z)\cdot m(o(e))\Big).\end{aligned}$$

Lemma 3.3 *For $\omega \in \mathrm{Hom}(\Gamma,\mathbb{R})$, let $\phi_t(x_0) = \phi_{t\omega}(x)_0$ and $\psi_t(x_0) = \psi_{t\omega}(x_0)$. For any $k \in \mathbb{N}$, changing eigensections s_ω, if necessary, for any $1 \le i \le k$, the i-th derivatives$\phi_o^{(i)}(x_0)$ and $\psi_0^{(i)}$ are the i-th order real coefficient homogeneous polynomials of $\sqrt{-1}\omega$. In particular,*

$$\phi_0(x_0) = |V_0|^{-1/2}, \quad \psi_0(x_0) = |V_0|^{1/2}m(x_0), \quad \phi_0'(x_0) = 0 \qquad (x_0 \in V_0).$$

Furthermore ψ_0' is a purely imaginary-valued first order polynomial of ω satisfying

$$\begin{cases}(I - {}^tL)\psi_0'(x_0) = 2\pi\sqrt{-1}|V_0|^{1/2}\Big(\sum_{e\in(E_0)_{x_0}} p(\overline{e})\omega(e)m(t(e)) \\ \qquad\qquad\qquad -m(x_0)\sum_{e\in E_0} p(\overline{e})\omega(e)m(t(e))\Big) \qquad (x_0\in V_0) \\ \sum_{z\in V_0}\psi_0'(z) = 0,\end{cases}$$

and ϕ_0'' and ψ_0'' are real-valued second oder polynomials of ω, satisfying

$$\begin{cases}(I - L)\phi_0''(x_0) = -4\pi^2|V_0|^{-1/2}\Big(\sum_{e\in(E_0)_{x_0}} p(e)\omega(e)^2 \\ \qquad\qquad\qquad -\sum_{e\in E_0} p(e)\omega(e)^2 m(o(e))\Big) \qquad (x_0\in V_0) \\ \sum_{z\in V_0}\phi_0''(z) = 0\end{cases}$$

and

$$\begin{cases}(I-{}^tL)\psi_0''(x_0)=4\pi\sqrt{-1}\Big(\sum_{e\in(E_0)_{x_0}}p(\overline{e})\omega(e)\psi_0'(t(e))+\langle\gamma_p,\omega\rangle\psi_0'(x_0)\Big)\\ \qquad\qquad -4\pi^2\Big(\sum_{e\in(E_0)_{x_0}}p(\overline{e})\omega(e)^2\psi_0(t(e))\\ \qquad\qquad\qquad -m(x_0)\sum_{e\in E_0}p(\overline{e})\omega(e)^2\psi_0(t(e))\Big) \qquad (x_0\in V_0)\\ \sum_{z\in V_0}\psi_0''(z)=-|V_0|\sum_{z\in V_0}\phi_0''(z)m(z),\end{cases}$$

respectively.

Applying the Fourier analysis, we conclude the following:

Theorem 3.4 ([1], [2]) *Let $x\in A_i$ and $y\in A_j$. If $n=Kl+j-i$,*

$$\begin{aligned}&(2\pi n)^{d/2}p(n,x,y)m(y)^{-1}\\ \sim\ & K\mathrm{vol}(\mathrm{Alb}^{\Gamma})e^{\left(-\frac{\left|\Phi_0(x)-\Phi_0(y)-n\rho_{\mathbb{R}}(\gamma_p)\right|^2_{g_0}}{2n}\right)}\Big(1+a_1(x,y;\gamma_p)n^{-1}\Big)\quad \text{as } n\to\infty.\end{aligned}$$

If $n\neq Kl+j-i$, then $p(n,x,y)=0$, where

$$\begin{aligned}&a_1\big(\pi(x),\pi(y),\gamma_p;\Phi_0(y)-\Phi_0(x)-n\rho_{\mathbb{R}}(\gamma_p)\big)\\ &=\frac{m(\pi(y))^{-1}}{2\pi}\sum_{i=1}^{d}\mathfrak{q}_i\big(\Phi_0(y)-\Phi_0(x)-n\rho_{\mathbb{R}}(\gamma_p)\big)_i\\ &\quad+\frac{1}{16\pi^3}\sum_{i,j=1}^{d}\mathfrak{q}_{iij}\big(\Phi_0(y)-\Phi_0(x)-n\rho_{\mathbb{R}}(\gamma_p)\big)_j-\frac{m(\pi(y))^{-1}}{8\pi^2}\sum_{i=1}^{d}\mathfrak{q}_{ii}\\ &\quad-\frac{m(\pi(y))^{-1}}{32\pi^4}\sum_{i,j=1}^{d}\mathfrak{q}_i\mathfrak{q}_{ijj}-\frac{1}{128\pi^4}\sum_{i,j=1}^{d}\mathfrak{q}_{iijj}-\frac{5}{1536\pi^6}\sum_{i,j,k=1}^{d}\mathfrak{q}_{iij}\mathfrak{q}_{jkk}\end{aligned}$$

with some coefficients $\mathfrak{q}_\alpha=\mathfrak{q}_\alpha(\pi(x),\pi(y);\gamma_p)$ $(\alpha=(\alpha_1,\dots,\alpha_r)\in\{1,\dots,d\}^r, r=1,\dots,4)$.

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